ABSORBING-REFLECTING FACTORIZATIONS FOR BIRTH-DEATH CHAINS ON THE INTEGERS AND THEIR DARBOUX TRANSFORMATIONS

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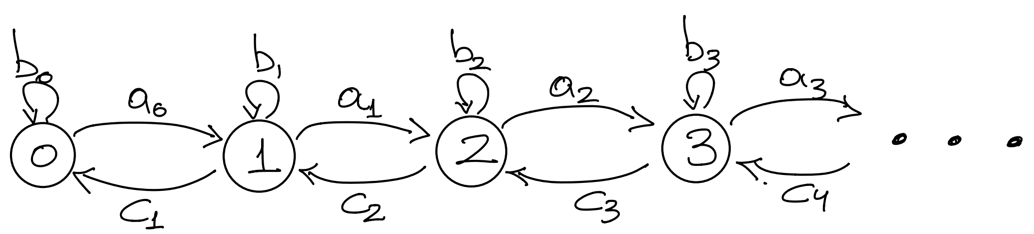
1) LU FACTORIZATIONS FOR BDC ON $\mathbb{N}_0$ (Grünbaum-dI, 2018)

Let $\{X_n, n=0,1,2,\dots\}$ be an irreducible discrete-time birth-death chain on the nonnegative integers $\mathbb{N}_0=\{0,1,2,\dots\}$.



The transition probability matrix is given by

$$P=\begin{pmatrix} b_0 & a_0 & 0 & 0 & \\ c_1 & b_1 & a_1 & 0 & \\ 0 & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \begin{array}{l} 0<a_n, c_{n+1}<1,\ 0\leq b_n<1,\ n\geq 0, \\ a_0+b_0=1, \\ a_n+c_n+b_n=1,\ n\geq 1 \end{array}$$

We look for STOCHASTIC UL (or LU) factorizations of P of the form.

$$P = \underbrace{\begin{pmatrix} y_0 & x_0 & 0 & 0 \\ 0 & y_1 & x_1 & 0 \\ 0 & 0 & y_2 & x_2 \\ & & & \ddots & \ddots \end{pmatrix}}_{P_U} \underbrace{\begin{pmatrix} s_0 & 0 \\ r_1 & s_1 & 0 \\ 0 & r_2 & s_2 & 0 \\ & \ddots & \ddots & \ddots \end{pmatrix}}_{P_L}$$

or $P = \tilde{P}_L \tilde{P}_U$, with $\tilde{x}_n, \tilde{y}_n, \tilde{s}_n, \tilde{r}_n \; n \geq 0$, where P_U and P_L are ALSO STOCHASTIC matrices.

THEOREM.

a) UL case: one free parameter (y_0). The stochastic UL fact. is possible if and only if

$$0 \leq y_0 \leq H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}}$$

if we assume that $0 < N_n < D_n$, where $h_n = N_n / D_n$, and $(h_n)_{n \in \mathbb{N}_0}$ are the convergents of H.

b) LU case: The (unique) stochastic LU fact is possible iff $H \geq 0$, with the same assumptions.

PROOF: Based on the relation between coefficients in $P = P_U P_L$:

$$\begin{cases} a_n = x_n s_{n+1}, \; n \geq 0 \\ b_n = x_n r_{n+1} + y_n s_n, \; n \geq 0 \\ c_n = y_n r_n, \; n \geq 1 \end{cases}$$

$$y_{n+1} = \frac{c_{n+1}}{1 - a_n/(1-y_n)}, \; n \geq 0.$$

$$s_{n+1} = \frac{a_n}{1 - c_n/(1-s_n)}, \; n \geq 1. \qquad \square$$

ORIGINAL BDC	DARBOUX TRANFORMATION
$P = P_u P_L \quad (P = \tilde{P}_L \tilde{P}_u)$	$\tilde{P} = P_L P_u \quad (\hat{P} = \tilde{P}_u \tilde{P}_L)$
Spectral or Favard's THM. $PQ = xQ, \quad Q = (Q_0, Q_1, \dots)^T$ P on $\ell^2_\pi(\mathbb{N}_0) \iff \varphi(x) \geq 0$ on $[-1,1]$ $\int_{-1}^{1} Q_n(x) Q_m(x) \varphi(x) dx = \pi_n^{-1} \delta_{n,m}$ $\pi_0 = 1, \quad \pi_n = \frac{a_0 a_1 \cdots a_{n-1}}{c_1 c_2 \cdots c_n}, \ n \geq 1.$	UL) $\tilde{Q} = P_L Q \Rightarrow \tilde{P}\tilde{Q} = x\tilde{Q}$ If $\mu_{-1} = \int_{-1}^{1} x^{-1} \varphi(x) dx < \infty$ $\Rightarrow \tilde{\varphi}(x) = y_0 \frac{\varphi(x)}{x} + (1 - y_0 \mu_{-1}) \delta_0(x)$ LU) $R = P_u Q \to \hat{Q} = \frac{1}{x} R \to \hat{P}\hat{Q} = x\hat{Q}$ $\Rightarrow \hat{\varphi}(x) = x \varphi(x)$
Karlin-McGregor representation $P_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i)$ $= \pi_j \int_{-1}^{1} Q_i(x) Q_j(x) \varphi(x) dx$	UL) $\tilde{P} \to \{\tilde{X}_n, n = 0, 1, 2, \dots\}$ $\tilde{P}_{ij}^{(n)} = \mathbb{P}(\tilde{X}_n = j \mid \tilde{X}_0 = i)$ $= \tilde{\pi}_j \int_{-1}^{1} \tilde{Q}_i(x) \tilde{Q}_j(x) \tilde{\varphi}(x) dx$ LU) $\hat{P} \to \{\hat{X}_n, n = 0, 1, 2, \dots\}$ $\hat{P}_{ij}^{(n)} = \mathbb{P}(\hat{X}_n = j \mid \hat{X}_0 = i)$ $= \hat{\pi}_j \int_{-1}^{1} \hat{Q}_i(x) \hat{Q}_j(x) \hat{\varphi}(x) dx$

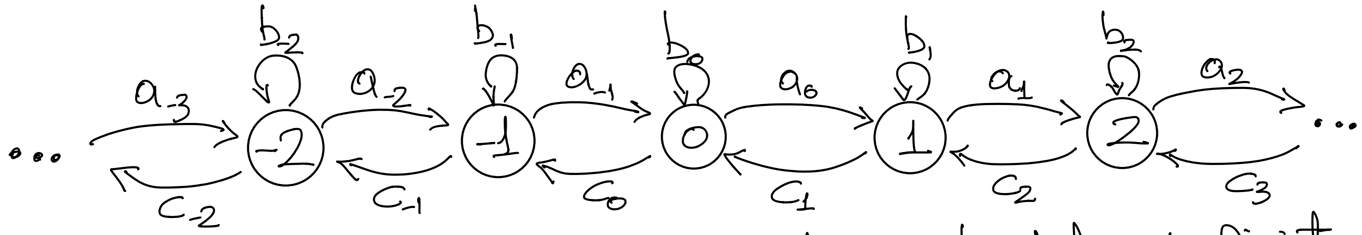
Example: Jacobi polynomials on $[0,1]$, $Q_n^{\alpha,\beta}(1) = 1$,

$P = P_u P_L \iff 0 \leq y_0 \leq \frac{\alpha}{\alpha+\beta+1}, \quad \alpha \geq 0, \ \beta > -1$

$\hookrightarrow$ For $y_0 = \frac{\alpha}{\alpha+\beta+1}$, this factorization allows for an easier urn model for the Jacobi polynomials

2) LU FACTORIZATIONS FOR BDC ON $\mathbb{Z}$ (dI-Juarez, 2020)

Let $\{X_n, n=0,1,2,\dots\}$ be an irreducible discrete-time BDC on $\mathbb{Z}$.

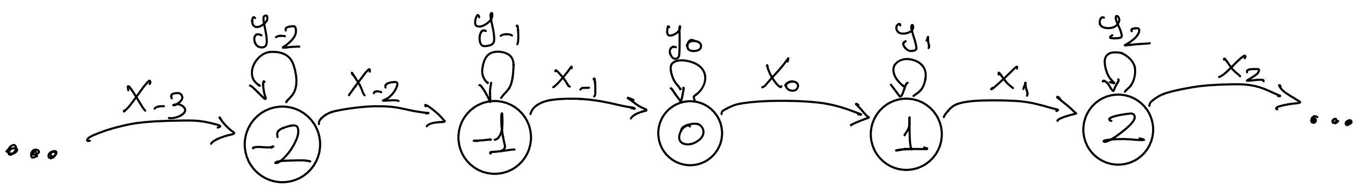


Now, the transition probability matrix is doubly infinite

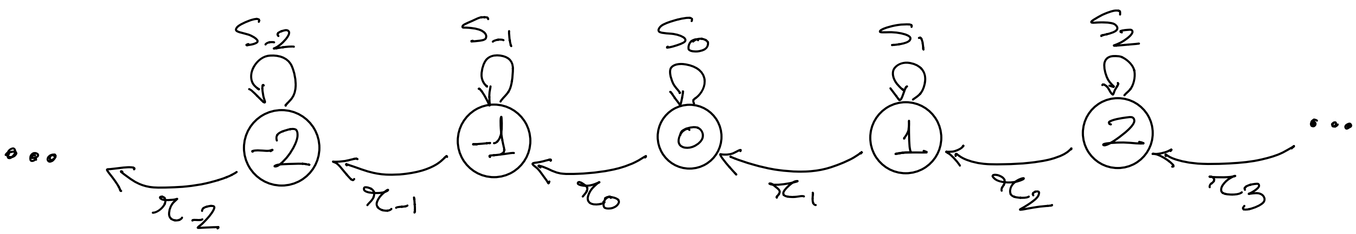
$$P = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ c_{-2} & b_{-2} & a_{-2} & 0 & & \\ 0 & c_{-1} & b_{-1} & a_{-1} & 0 & \\ \hline & & c_0 & b_0 & a_0 & 0 \\ & & & c_1 & b_1 & a_1 \\ & & & 0 & c_2 & b_2 \quad a_2 \\ & & & & \ddots & \ddots \quad \ddots \end{array}\right), \quad \begin{array}{l} 0 < a_n, c_n < 1, \; n \in \mathbb{Z} \\ a_n + b_n + c_n = 1, \; n \in \mathbb{Z} \end{array}$$

Again, we look for STOCHASTIC UL (or LU) factorizations of P of the form $P = \underbrace{P_U P_L}$ (or $P = \tilde{P}_L \tilde{P}_U$).

P_U represents a pure-birth chain on $\mathbb{Z}$ with diagram



P_L represents a pure-death chain on $\mathbb{Z}$ with diagram



Now we need TWO continued fractions:

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}}, \quad H' = \cfrac{c_0}{1 - \cfrac{a_{-1}}{1 - \cfrac{c_{-1}}{1 - \cfrac{a_{-2}}{1 - \cdots}}}}$$

with convergents $h_n = N_n / D_n$ and $h'_{-n} = N'_{-n} / D'_{-n}$

THEOREM.

a) UL case: one free parameter (y_0). Assuming $0 < N_n < D_n$, $0 < N'_{-n} < D'_{-n}$ and $\boxed{H' \leq H}$ the stochastic UL fact. is possible if and only if

$$H' \leq y_0 \leq H$$

b) LU case: now one free parameter also ($\tilde{r}_0$). With the same assumptions as before now we need

$$\boxed{H' \leq \tilde{r}_0 \leq H}$$

SPECTRAL ANALYSIS OF P (Nikishin, 1986 or Masson-Repka, 1991) ←

The eigenvalue equation $PQ^\alpha = x Q^\alpha$, $Q^\alpha = (\ldots, Q_{-1}^\alpha, Q_0^\alpha, Q_1^\alpha, \ldots)$ now gives TWO polynomial families of l.i. solutions $Q_n^\alpha, \alpha = 1,2$

$$\begin{cases} x Q_n^\alpha = a_n Q_{n+1}^\alpha + b_n Q_n^\alpha + c_n Q_{n-1}^\alpha, \; n \in \mathbb{Z}, \alpha = 1,2 \\ Q_0^1(x) = 1, \quad Q_{-1}^1(x) = 0 \\ Q_0^2(x) = 0, \quad Q_{-1}^2(x) = 1 \end{cases}$$

It is easy to see that:

$$\deg(Q_n^1) = n,\ n \geq 0, \quad \deg(Q_n^2) = n-1,\ n \geq 1$$

$$\deg(Q_{-n-1}^1) = n-1,\ n \geq 1, \quad \deg(Q_{-n-1}^2) = n,\ n \geq 0$$

(*) P is selfadjoint in $\ell^2_\pi(\mathbb{Z})$, where $(\pi_n)_{n\in\mathbb{Z}}$ is given by

$$\pi_0 = 1, \quad \pi_n = \frac{a_0 \cdots a_{n-1}}{c_1 \cdots c_n}, \quad \pi_{-n} = \frac{c_0 c_{-1} \cdots c_{-n+1}}{a_{-1} a_{-2} \cdots a_{-n}}, \quad n \geq 1$$

It is possible to see that

$$Q_i^1(P)e^{(0)} + Q_i^2(P)e^{(-1)} = e^{(i)}, \ i \in \mathbb{Z}, \quad \underline{e^{(i)} = \left(\delta_{ij}/\pi_i\right)_{j\in\mathbb{Z}}}$$

We need to apply the Spectral Theorem 4 times and obtain 3 measures $\psi_{\alpha,\beta}$, $\alpha = 1,2$ (since $\psi_{12} = \psi_{21}$ by the symmetry) such that $\psi_{11}, \psi_{22} \geq 0$

$$\boxed{\sum_{\alpha,\beta=1}^{2} \int_{-1}^{1} Q_i^\alpha(x) Q_j^\beta(x)\, d\psi_{\alpha,\beta}(x) = \frac{\delta_{ij}}{\pi_j}, \quad i,j \in \mathbb{Z}}$$

The 2×2 matrix of measures

$$\Psi(x) = \begin{pmatrix} \psi_{11}(x) & \psi_{12}(x) \\ \psi_{12}(x) & \psi_{22}(x) \end{pmatrix}$$

is called the spectral matrix associated with P.

REMARK: There is a natural connection with the theory of matrix-valued orthogonal polynomials. If we define

$$Q_n(x) = \begin{pmatrix} Q_n^1(x) & Q_n^2(x) \\ Q_{-n-1}^1(x) & Q_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0,$$

$$\deg Q_n = n.$$

then we have

$$\begin{cases} x\,Q_0(x) = A_0 Q_1(x) + B_0 Q_0(x), & Q_0(x) = I_{2\times 2} \\ x\,Q_n = A_n Q_{n+1} + B_n Q_n + C_n Q_{n-1}, & n \geq 1 \end{cases}$$

where

$$B_0 = \begin{pmatrix} b_0 & c_0 \\ a_{-1} & b_{-1} \end{pmatrix}, \quad B_n = \begin{pmatrix} b_n & 0 \\ 0 & b_{-n-1} \end{pmatrix}, n \geq 1$$

$$A_n = \begin{pmatrix} a_n & 0 \\ 0 & c_{-n-1} \end{pmatrix}, n \geq 0, \quad C_n = \begin{pmatrix} c_n & 0 \\ 0 & a_{-n-1} \end{pmatrix}, n \geq 1$$

and

$$\boxed{\int_{-1}^{1} Q_n(x)\, d\Psi(x)\, Q_m^T(x) = \begin{pmatrix} 1/\pi_n & 0 \\ 0 & 1/\pi_{-n-1} \end{pmatrix} \delta_{n,m}}$$

The transition probability matrix is block tridiagonal

$$\mathbf{P} = \begin{pmatrix} B_0 & A_0 & 0 & \\ C_1 & B_1 & A_1 & \\ 0 & C_2 & B_2 & A_2 \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

"folding trick" (Berezanski, 1968)

This is a.k.a. a quasi-birth-death process on $\mathbb{N} \times \{1,2\}$.

Darboux transformation

UL) $\tilde{P} = P_L P_U$ on $\mathbb{Z}$. The spectral matrix for $\tilde{P}$ is

$$\boxed{\tilde{\Psi}(x) = S_0(x)\,\Psi_S(x)\,S_0^*(x)}$$

where

$$S_0(x) = \begin{pmatrix} s_0 & r_0 \\ -\frac{x_{-1} s_0}{y_{-1}} & \frac{x - x_{-1} r_0}{y_0} \end{pmatrix}$$

and

$$\boxed{\Psi_S(x) = \frac{y_0}{s_0}\frac{\Psi(x)}{x} + \left[\begin{pmatrix} 1/s_0 & 0 \\ 0 & 1/r_0 \end{pmatrix} - \frac{y_0}{s_0} M_{-1}\right]\delta_0(x)}$$

where $M_{-1} = \int_{-1}^{1} x^{-1}\Psi(x)dx < \infty$ (assumed). $\leftarrow$

UL) $\hat{P} = \tilde{P}_U \tilde{P}_L$ on $\mathbb{Z}$. The spectral matrix for $\hat{P}$ is

$$\boxed{\hat{\Psi}(x) = T_0(x)\Psi_T(x)T_0^*(x)}$$

where

$$T_0(x) = \begin{pmatrix} \frac{x - \tilde{r}_0\tilde{x}_{-1}}{\tilde{s}_0} & -\tilde{r}_0\tilde{y}_{-1} \\ \tilde{x}_{-1} & \tilde{y}_{-1} \end{pmatrix}$$

and

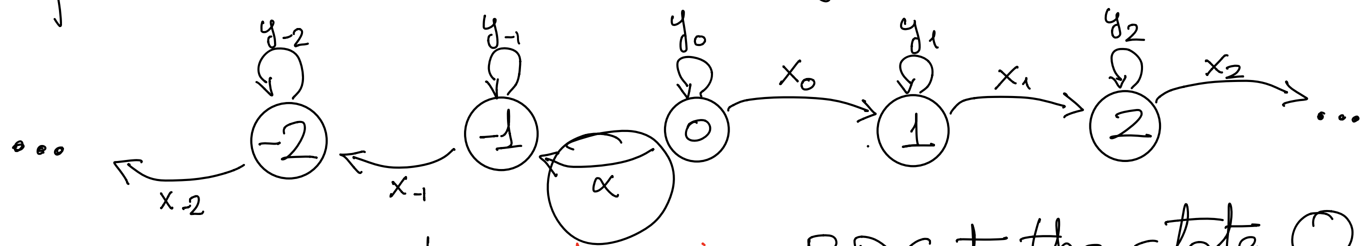
$$\boxed{\Psi_T(x) = \frac{\tilde{s}_0}{\tilde{y}_0}\frac{\Psi(x)}{x} + \left[\frac{\hat{a}_{-1}}{\tilde{r}_0}\begin{pmatrix} 1/\tilde{x}_{-1} & 0 \\ 0 & 1/\tilde{y}_{-1} \end{pmatrix} - \frac{\tilde{s}_0}{\tilde{y}_0} M_{-1}\right]\delta_0(x)}$$

$\circledast$ In [dI-Juarez, 2020] we studied 2 examples ith constant transition probabilities.

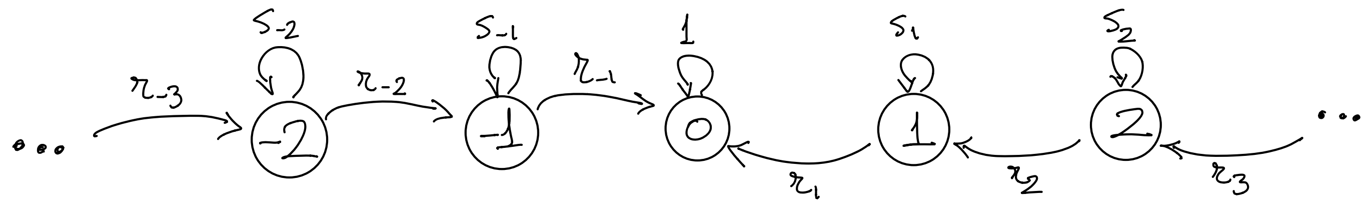
3) AR FACTORIZATIONS FOR BDC ON $\mathbb{Z}$ (dI-Juarez, 2021)

Let $\{X_n, n=0,1,2,\dots\}$ be an irreducible discrete-time BDC on $\mathbb{Z}$ with transition probability matrix P (doubly stochastic). In 2) we considered UL (or LU) factorizations $P = P_U P_L$. P_U represents a pure-birth process ($\to\infty$) while P_L represents a pure-death process ($\to -\infty$).

Now we consider a new factorization of the form $\boxed{P = P_R P_A}$, where P_R represents a REFLECTING BDC from the state 0 with diagram



and P_A represents an ABSORBING BDC to the state 0:



REMARKS

1) If the state space is $\mathbb{N}_0$, then AR factorizations are LU.

2) We need to introduce a new parameter α to connect the reflecting BDC from the state 0 to the state -1.

3) $P = P_R P_A$ does not preserve the UL or LU structure, but, after the "folding trick", if we write $\boxed{\mathbf{P} = \mathbf{P}_R \mathbf{P}_A}$, then we have

$$\mathbf{P}_R = \left(\begin{array}{cc|cc|cc} y_0 & \alpha & x_0 & 0 & & \\ 0 & y_{-1} & 0 & x_{-1} & & \\ \hline & & y_1 & 0 & x_1 & 0 \\ & & 0 & y_{-2} & 0 & x_{-2} \\ \hline & & & & \ddots & \ddots \end{array}\right), \quad \mathbf{P}_A = \left(\begin{array}{cc|cc|c} 1 & 0 & & & \\ r_{-1} & s_{-1} & & & \\ \hline r_1 & 0 & s_1 & 0 & \\ 0 & r_{-2} & 0 & s_{-2} & \\ \hline & & \ddots & & \ddots \end{array}\right)$$

which preserves the block UL structure

In the case of $P = P_U P_L$, P_U is upper bidiagonal and P_L is lower bidiagonal, but, after the "folding", $\boxed{\mathbf{P} = \mathbf{P}_U \mathbf{P}_L}$, $\mathbf{P}_U$ is NOT upper block bidiagonal and $\mathbf{P}_L$ is NOT lower block bidiagonal

Again, if we define

$$H = \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}} \, , \quad H' = \cfrac{c_0}{1 - \cfrac{a_{-1}}{1 - \cfrac{c_{-1}}{1 - \cfrac{a_{-2}}{1 - \cdots}}}}$$

with convergents $h_n = N_n / D_n$ and $h'_{-n} = N'_{-n} / D'_{-n}$

THEOREM

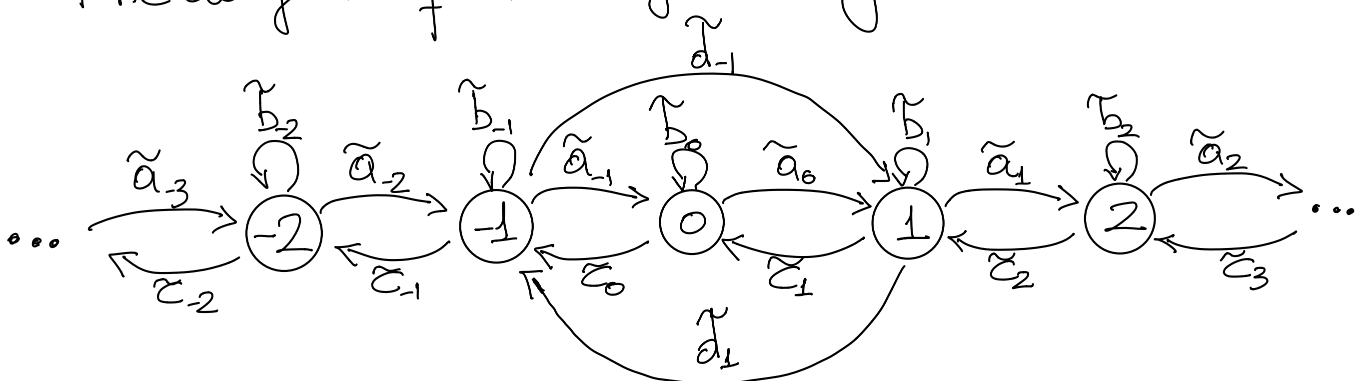
We have now TWO free parameters (α, x_0). Assuming $0 < N_n < D_n$, $0 < N'_{-n} < D'_{-n}$ and $H + H' \leq 1$, the stochastic RA fact. is possible if and only if

$$\boxed{\alpha \geq H', \quad x_0 \geq H}$$

Darboux transformation

$\tilde{P} = P_A P_R$ is NOT a BDC since now we have extra transitions between the states -1 and 1.

The diagram of $\tilde{P}$ is given by



Nevertheless, using the UL block factorization of $\tilde{P}$ after the "folding", we can compute the spectral matrix $\tilde{\Psi}(x)$, given by

$$\boxed{\widetilde{\Psi}(x) = S_0 \Psi_u(x) S_0^T,} \qquad S_0 = \begin{pmatrix} 1 & 0 \\ r_{-1} & s_{-1} \end{pmatrix}$$

and

$$\boxed{\Psi_u(x) = \frac{\Psi(x)}{x} + \left[\frac{1}{y_0} \begin{pmatrix} 1 & -r_{-1}/s_{-1} \\ -r_{-1}/s_{-1} & (r_{-1}/s_{-1})^2 + y_0 r_{-1}/x s_{-1}^2 \end{pmatrix} - M_{-1} \right] S_0(x)}$$

where $M_{-1} = \int_{-1}^{1} x^{-1} \Psi(x) dx < \infty$ (assumed).

For the absorbing-reflecting (AR) factorization of $P = \widetilde{P}_A \widetilde{P}_R$, now we need to start from a Markov chain P with extra transitions between the states -1 and 1 (otherwise $\widetilde{P}_A$ and $\widetilde{P}_R$ have to be splitted into two separated BDC at 0).

THEOREM.
The AR stochastic factorization is now unique and possible if and only $d_{-1} = \frac{a_{-1}a_0}{b_0}$, $d_1 = \frac{c_0 c_1}{b_0}$ and

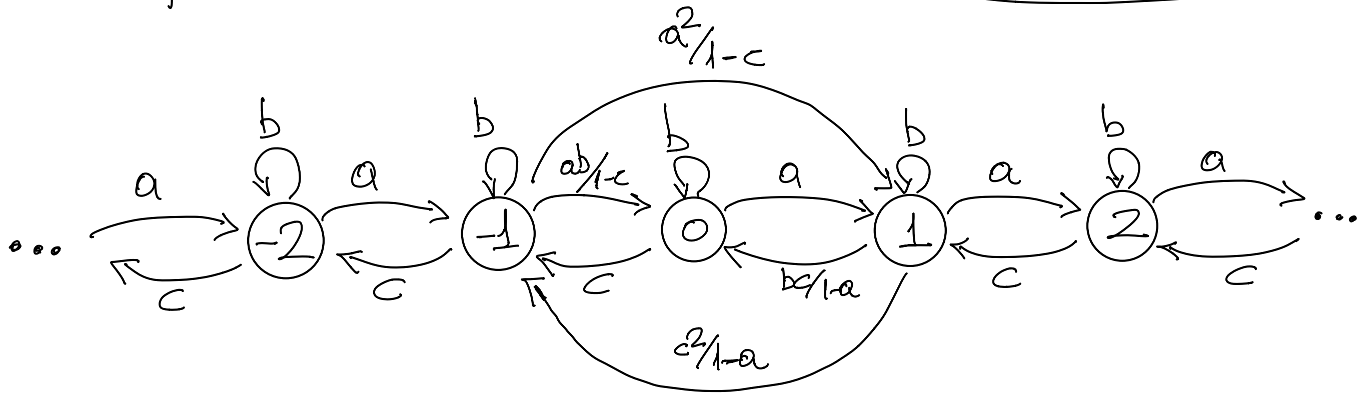
$$b_0 \geq \min\left\{ \frac{a_0}{H}, \frac{c_0}{H'} \right\}$$

(with the same assumptions on H, H').

As for the Darboux transformation $\widehat{P} = \widetilde{P}_R \widetilde{P}_A$ (NOW TRIDIAGONAL) the spectral matrix is given by

$$\boxed{\widehat{\Psi}(x) = x \widetilde{S}_0^{-1} \Psi(x) \widetilde{S}_0^{-T},} \qquad \widetilde{S}_0 = \begin{pmatrix} 1 & 0 \\ \widetilde{r}_{-1} & \widetilde{s}_{-1} \end{pmatrix}.$$

Example: $a_n = a,\ n\in\mathbb{Z}\setminus\{-1\}$, $b_n = b,\ n\in\mathbb{Z}$, $c_n = c,\ n\in\mathbb{Z}\setminus\{1\}$



- It is possible to see that the AR factorization is always possible.
- We can compute the spectral matrix $\Psi(x)$ of P using tools of MVOPs. The case $\boxed{a=c}$ is simpler and $\Psi(x)$ has only an absolutely continuous part

$$\boxed{\Psi(x) = \frac{\sqrt{(1-x)(x-1+4a)}}{c\pi\, p(x)}\begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{12}(x) & q_{22}(x)\end{pmatrix},\quad x\in[1-4a,1]}$$

$$\begin{cases} q_{11}(x) = -2(1-2a)(1-a)\big(x(1-a)+a\big), \\ q_{12}(x) = \frac{1}{2}q_{11}(x)(x-1+2a), \\ q_{22}(x) = -(1-2a)(1-a)\left[x^2-2(1-a)(1-2a)x+(1-2a)^2\right], \\ p(x) = 2(1-x)\big(x(1-a)+a\big)\left[(a-1)x^2+(1-2a)^2x-a(1-2a)^2\right]. \end{cases}$$

- The Darboux transformation is a highly nontrivial BDC on $\mathbb{Z}$ but we know the spectral matrix given by

$$\boxed{\hat{\Psi}(x) = x\tilde{S}_0^{-1}\Psi(x)\tilde{S}_0^{-T},}\qquad \tilde{S}_0^{-1} = \begin{pmatrix} 1 & 0 \\ -a/b & (1-c)/b\end{pmatrix}.$$